where

$$\beta^2 = \frac{P}{[1 - (P/kA_{55})]D_c} \tag{22}$$

The general solution of Eq. (21) is

$$w(x) = C_2 \cos \beta x + C_4 + \frac{q}{2P} x^2 \tag{23}$$

The three unknowns, P, C_2, and C_4, can be determined from the boundary conditions, Eqs. (15–17), in a similar way as was the case of $N_x > 0$.

Transition State with Vanishing In-Plane Force ($N_x = 0$)

The state corresponding to $N_x = 0$ is referred to as the "transition state," which implies that the induced in-plane force changes its sign when this state is passed. For this case, the governing equation (11) with a uniform transverse load q_t can be expressed as

$$w_{,xxxx} = \frac{q_t}{D_c} \tag{24}$$

The general solution of Eq. (24) is

$$w = C_2 x^2 + C_4 + \frac{q_t}{24D_c} x^4 \tag{25}$$

The three unknowns, C_2, C_4, and q_t, are determined from the boundary conditions Eqs. (15–17). The resulting transverse deflection w and transverse load q_t are

$$w = \frac{q_t}{24D_c}(5a^2 - x^2)(a^2 - x^2) + \frac{q_t}{2kA_{55}}(a^2 - x^2) \tag{26}$$

$$q_t = -\frac{B_{11}}{A_{11}}\left[\frac{17a^4}{210D_c} + \frac{2a^2}{5kA_{55}} + \frac{D_c}{2(kA_{55})^2}\right]^{-1} \tag{27}$$

It is noted from Eq. (27) that q_t must be in opposite sign as the coupling stiffness B_{11}. Therefore, a transition state can only exist when B_{11} and q are in opposite signs.

Results

Numerical solutions are obtained based on the following graphite/epoxy composite properties: $E_1 = 20 \times 10^6$ psi, $E_2 = 1.4 \times 10^6$ psi, $G_{12} = G_{13} = 0.8 \times 10^6$ psi, $G_{23} = 0.6 \times 10^6$ psi, $\nu_{12} = 0.3$. According to Ref. 4, the shear correction factor k is taken as $\pi^2/12$. The ply thickness is 0.005 in. For the laminate $[90_4/0_4]$, Figs. 1 and 2 show the comparison between the nonlinear solutions and the linear solutions. The induced in-plane force and maximum deflection predicted by the linear theories (LSDT and LCLT) are proportional to the transverse load. However, the nonlinear theories (NSDT and NCLT) give quite different results. As shown in Fig. 1, negative q induces positive in-plane force N_x with a negative coupling stiffness B_{11}. As discussed earlier, a transition state exists only when B_{11} and q are in the opposite signs. As a result, positive q will induce a compressive in-plane force initially, which turns into tension once q passes the transition transverse load q_t.

Figure 2 shows that the maximum deflections predicted by nonlinear theories are also significantly different from those predicted by linear theories. In addition, the results from NSDT and NCLT also show that the deflection with positive q is larger than that with negative q. This is attributed to the compressive in-plane force initially induced by positive q, which tends to aggravate the transverse deflection. Figure 2 also shows that the transverse shear effects are less significant from the nonlinear theories for the present case.

Theoretically, the transition state exists for every unsymmetric laminate as long as the transverse load q and the laminate coupling stiffness B_{11} have opposite signs. Figure 3 shows the variation of the transition transverse load q_t, obtained from Eq. (27), for laminates with slenderness ratios L/h up to 100. For states above this curve, the induced in-plane forces are in tension. For those below this curve, the induced in-plane forces are in compression. Note that q_t varies drastically with the slenderness ratio. For a thin laminate with rather large slenderness ratio, q_t becomes negligibly small. Since the transverse shear effect is more important for thick plates, it is noted in Fig. 3 that the transverse shear effect on q_t becomes increasingly more pronounced as L/h is less than 20.

References

[1]Sun, C. T., and Chin, H., "Analysis of Asymmetric Composite Laminates," *AIAA Journal*, Vol. 26, No. 6, 1988, pp. 714–718.

[2]Sun, C. T., and Chin, H., "On Large Deflection Effects in Unsymmetric Cross-Ply Composite Laminates," *Journal of Composite Materials*, Vol. 22, No. 11, 1988, pp. 1045–1059.

[3]Chen, H. P., "Shear Deformation Theory for Compressive Delamination Buckling and Growth," *AIAA Journal*, Vol. 29, No. 5, 1991, pp. 813–819.

[4]Whitney, J. M., and Sun, C. T., "A Higher Order Theory for Extensional Motion of Laminated Composites," *Journal of Sound and Vibration*, Vol. 30, No. 1, 1973, pp. 85–97.

Matrix Transformation Method for Updating Dynamic Model

De-Wen Zhang*
Beijing Institute of Structure and Environment, Beijing, People's Republic of China
and
Lingmi Zhang†
Nanjing Aeronautical Institute, Nanjing, People's Republic of China

Introduction

Many systematic methods[1-13] have been developed in recent years for updating analytical models to predict modal test data more closely. The methods in Refs. 1–7, referred to as matrix-type procedures here, correct the whole mass and stiffness matrices. Correspondingly, the methods in Refs. 8–13, element-type procedures, modify only some nonzero elements of mass and stiffness matrices. In the former procedures, the connectivity of the original analytical model is not preserved, causing the addition of unwanted load paths. In the latter, particularly in Refs. 9 and 11, the stiffness matrix may be identified exactly in certain cases even when some of the test modes are not known.

The purpose of the work presented in this Note is similar to that of Ref. 8. This Note proposes a matrix transform method (MTM), in which the derivations are much simpler than those in Refs. 1–6, which use Lagrange multipliers. In the present MTM, transform matrices for the correction of dynamic models are created. The effect of the transformation is opposite to that of the objective function in the Lagrange multiplier method (LMM); the former is to reduce the number of unknown parameters of the governing equations and the latter is to increase the number of equations. Not only can MTM reproduce the formulas in Refs. 1–6, but it can also derive some new formulations that are difficult to be formed by LMM.

Received Oct. 29, 1990; revision received April 22, 1991; accepted for publication May 9, 1991.

*Senior Research Engineer, P.O. Box 9210. Member AIAA.
†Professor of Vibration Engineering, Box 109.

Mathematical Background

In this section, the minimum weighted norm solutions for two kinds of indeterminate equations are described.

Basic Formulas

Let

$$X = BYB^t \tag{1}$$

$$D = A^tEA \tag{2}$$

where $X, E \in R^{n,n}$; $A, B \in R^{n,r}$; $Y, D \in R^{r,r}$; $r < n$. One can find a matrix $C \in R^{r,r}$, which satisfies in the sense of least squares

$$B = AC \tag{3}$$

Substituting Eq. (3) into Eq. (1) yields

$$X = A\tilde{Y}A^t \tag{4}$$

where

$$\tilde{Y} = CYC^t \tag{5}$$

Obviously, the matrix C in Eq. (3) is the minimum norm solution in the sense of $\|AC-B\| = \min$, where $\|.\|$ represents the Euclidean norm. It can be known by Eqs. (2) and (4) that

$$x_{ij} = \sum_{p=1}^{r}\sum_{q=1}^{r} a_{ip}a_{jq}\tilde{y}_{pq}, \qquad (i,j = 1,2, \ldots, n) \tag{6}$$

$$d_{pq} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ip}a_{jq}e_{ij}, \qquad (p,q = 1,2, \ldots, r) \tag{7}$$

where $x_{ij}, e_{ij}, \tilde{y}_{pq}, d_{pq}, a_{ip}$, and a_{jq} are the elements of matrices $X, E, \tilde{Y}, D$, and A, respectively. Letting $x_j, e_j, \tilde{y}_q$, and d_q be the jth and qth column of matrices $X, E, \tilde{Y}$, and D, respectively, it can be shown that

$$\sum_{j=1}^{n} x_j^t e_j = \sum_{q=1}^{r} \tilde{y}_q^t d_q \tag{8}$$

Since

$$x_j^t e_j = \sum_{i=1}^{n} x_{ij}e_{ij}, \qquad \tilde{y}_q^t d_q = \sum_{p=1}^{r} \tilde{y}_{pq}d_{pq}$$

Eq. (8) will be reached directly by using Eqs. (6) and (7).

First kind of indeterminate equation:

$$A^tXA = L \qquad (L \in R^{r,r}) \tag{9}$$

Let

$$X = WBYB^tW \tag{10}$$

where $W \in R^{n,n}$ is a symmetric positive definite matrix. And letting $\bar{A} = W^{1/2}A$, $\bar{B} = W^{1/2}B$, and $\bar{X} = W^{-1/2}XW^{-1/2}$, Eqs. (9) and (10) are rewritten as

$$\bar{A}^t\bar{X}\bar{A} = L \tag{11}$$

$$\bar{X} = \bar{B}Y\bar{B}^t \tag{12}$$

In addition, it is shown by Eq. (3) that

$$\bar{B} = \bar{A}C \tag{13}$$

Embedding Eq. (13) into Eq. (12) yields

$$\bar{X} = \bar{A}\tilde{Y}\bar{A}^t \tag{14}$$

However, the general solution of Eq. (11) can be expressed as

$$Z = \bar{X} + E \tag{15}$$

Substituting Eq. (15) into Eq. (11) has

$$\bar{A}^tE\bar{A} = 0 \tag{16}$$

The Euclidean norm of Z is

$$\|Z\| = \|\bar{X}\| + \|E\| + 2\sum_{j=1}^{n} \bar{x}_j^t e_j \cdot 0 = 0 \tag{17}$$

in which x_j is the jth column of $\bar{X}$. Comparing Eqs. (14) and (16) with Eqs. (4) and (2), respectively, the following relationship can be generated from Eq. (8)

$$\sum_j^n \bar{x}_j^t e_j = \sum_q^r \tilde{y}_q^t d_q = \sum_q^r \tilde{y}_q^t \cdot 0 = 0 \tag{18}$$

Therefore,

$$\|Z\| = \|\bar{X}\| + \|E\| \geq \|\bar{X}\| \tag{19}$$

Now, embedding Eq. (12) into Eq. (11) allows the solution for $\bar{X}$:

$$\bar{X} = \bar{B}(\bar{A}^t\bar{B})^{-1}L(\bar{B}^t\bar{A})^{-1}\bar{B}^t \tag{20}$$

Thus, the following formula is obtained

$$X = WB(A^tWB)^{-1}L(B^tWA)^{-1}B^tW \tag{21}$$

When $B = A$, from Eq. (21), one has

$$X = WA(A^tWA)^{-1}L(A^tWA)^{-1}A^tW \tag{22}$$

From Eqs. (19) and (13), one knows that Eq. (22) is the minimum weighted norm solution for X in the sense of $\|W^{-1/2}XW^{-1/2}\| = \min$, and Eq. (21) is the solution of Eq. (9) in the sense of $\|W^{-1/2}XW^{-1/2}\| = \min$ and $\|\bar{A}C-\bar{B}\| = \|W^{1/2}(AC-B)\| = \min$.

Second kind of indeterminate equation:

$$A^tX = G \qquad (G \in R^{r,n}) \tag{23}$$

Similarly, Eq. (23) is rewritten as

$$\bar{A}^t\bar{X} = \bar{G} \tag{24}$$

in which $\bar{G} = GW^{-1/2}$. Here, $\bar{X}$ is taken as Eq. (12). Thus, embedding Eq. (12) into Eq. (24) yields

$$\bar{X} = \bar{B}(\bar{A}^t\bar{B})^{-1}\bar{G} \tag{25}$$

Therefore,

$$X = WB(A^tWB)^{-1}G \tag{26}$$

When $B = A$, Eq. (26) becomes

$$X = WA(A^tWA)^{-1}G \tag{27}$$

In the same manner, Eq. (26) is the solution for X in the sense of $\|W^{-1/2}XW^{-1/2}\| = \min$ and $\|W^{1/2}(AC-B)\| = \min$, and Eq. (27) is the solution for X only in the sense of $\|W^{-1/2}XW^{-1/2}\| = \min$.

Matrix Transform Method

Supposing the updated mass and stiffness matrices M and K are

$$M = M_a + \Delta M \tag{28}$$

$$K = K_a + \Delta K \tag{29}$$

namely, $\Delta M = M - M_a$ and $\Delta K = K - K_a$, where M_a and K_a are the mass and stiffness matrices of the initial finite element model (FEM), respectively. Assume that the updated model satisfies the orthogonality condition and eigen-equation

$$\Phi^t M \Phi = I \tag{30}$$

$$K\Phi = M\Phi\Lambda \tag{31}$$

where Λ and Φ are the test measured eigenpairs.

Reproduction of the Results in Refs. 1–3

Updating the Mass Matrix

Substituting Eq. (28) into Eq. (30) reaches

$$\Phi^t \Delta M \Phi = I - m_a \tag{32}$$

where $m_a = \Phi^t M_a \Phi$. In Refs. 1 and 2, the updated mass and stiffness matrices are weighted with the mass matrix M_a of the FEM. According to this and the principle of MTM, let the transformation be

$$\Delta M = M_a \Phi \Delta\bar{M} \Phi^t M_a \tag{33}$$

Eq. (33) means that $W = M_a$, $B = A = \Phi$, and $\Delta\bar{M} = Y$ [see Eq. (10)]. Embedding Eq. (33) into Eq. (32) reaches

$$M = M_a + M_a \Phi m_a^{-1} (I - m_a) m_a^{-1} \Phi^t M_a \tag{34}$$

Updating the Stiffness Matrix

Embedding Eq. (29) into Eq. (31) yields

$$\Delta K \Phi = M\Phi\Lambda - K_a \Phi \tag{35}$$

Let

$$\Delta K = M_a \Phi \Delta\bar{K} \Phi^t M_a \tag{36}$$

Substituting Eq. (36) into Eq. (35), one obtains

$$\Delta K = (M\Phi\Lambda - K_a\Phi) m_a^{-1} \Phi^t M_a = \Delta K_{ns} \tag{37}$$

The matrix ΔK_{ns} in Eq. (37) is unsymmetric. In order to find a symmetric metric matrix ΔK, supposing

$$\Delta K = \Delta K_{ns} + \Delta K_{ns}^t + P \tag{38}$$

Substituting Eq. (38) into Eq. (35) generates

$$P\Phi = M\Phi\Lambda - K_a\Phi - (\Delta K_{ns} + \Delta K_{ns}^t)\Phi \tag{39}$$

Comparing Eq. (39) with Eq. (35), from Eq. (37), it is seen that

$$P = [M\Phi\Lambda - K_a\Phi - (\Delta K_{ns} + \Delta K_{ns}^t)\Phi] m_a^{-1} \Phi^t M_a \tag{40}$$

Finally, embedding Eqs. (37) and (40) into Eq. (38), a symmetric matrix ΔK can be obtained, and one has, from Eq. (29),

$$K = K_a + S + S^t \tag{41}$$

$$S = (M\Phi\Lambda - K_a\Phi) m_a^{-1} \Phi^t M_a - ½ M_a \Phi m_a^{-1} (\Lambda - k_a) m_a^{-1} \Phi^t M_a \tag{42}$$

where $k_a = \Phi^t K_a \Phi$.

When M_a in Eq. (33) is replaced by M, matrix S in Eqs. (41) and (42) becomes

$$S = (M\Phi\Lambda - K_a\Phi)\Phi^t M - ½ M\Phi(\Lambda - k_a)\Phi^t M \tag{43}$$

Equation (43) is the result derived by Ref. 3. Note that the formulas in Refs. 4–6 can also be reproduced by MTM.

New Formulas Corresponding to the Methods of Refs. 1–3

Updating the Mass Matrix

By using MTM, some new formulas of modified mass matrix are produced. For example, we can build a transform matrix as follows,

$$\Delta M = M_a \Phi_a \Delta\bar{M} \Phi_a^t M_a \tag{44}$$

Substituting Eq. (44) into Eq. (32), a new formula of updated mass matrix can be derived as

$$M = M_a + M_a \Phi_a (\Phi^t M_a \Phi_a)^{-1} (I - m_a)(\Phi_a^t M_a \Phi)^{-1} \Phi_a^t M_a \tag{45}$$

Updating the Stiffness Matrix

Using MTM, some new corrected stiffness matrices are formulated. For example, supposing

$$\Delta K = M_a \Phi_a \Delta\bar{K} \Phi_a^t M_a \tag{46}$$

Embedding Eq. (46) into Eq. (35), then, in the same procedure as in Eqs. (36–40), we obtain a new formula for matrix S in Eq. (41)

$$S = (M\Phi\Lambda - K_a\Phi)(\Phi_a^t M_a \Phi)^{-1} \Phi_a^t M_a - ½ M_a \Phi_a (\Phi^t M_a \Phi_a)^{-1} (\Lambda - k_a)(\Phi_a^t M_a \Phi)^{-1} \Phi_a^t M_a \tag{47}$$

Finally, it is seen from Eqs. (27) and (26), that Eq. (41) together with Eq. (42) is the minimum weighted norm solution with $\|M_a^{-½} K M_a^{-½}\| = \min$, and Eq. (41) together with Eq. (47) is the solution with $\|M_a^{-½} K M_a^{-½}\| = \min$ and $\|M_a^{½}(\Phi C - \Phi_a)\| = \min$. From this, it is concluded that the solutions obtained with taking $B = A$ in Eq. (10), or $\Phi_a = \Phi$ in Eqs. (44) and (46), are better than those with $B \neq A$ or $\Phi_a \neq \Phi$.

Conclusions

Using the MTM described in this Note, we have reproduced the results of Refs. 1–6, which are formulated by LMM. Many new results can also be deduced that would be difficult to be produced by the methods in Refs. 1–6. A typical example was derived.

References

[1]Berman, A., "Mass Matrix Correction Using an Incomplete Set of Measured Modes," *AIAA Journal,* Vol. 17, No. 10, 1979, pp. 1147–1148.

[2]Wei, F. S., "Stiffness Matrix Correction from Incomplete Test Data," *AIAA Journal,* Vol. 18, No. 10, 1980, pp. 1274, 1275.

[3]Berman, A., and Wei, F. S., "Automated Dynamic Analytical Model Improvement," NASA CR-3452, July 1981.

[4]Baruch, M., "Correction of Stiffness Matrix Using Vibration Tests," *AIAA Journal,* Vol. 20, No. 3, 1982, pp. 441, 442.

[5]Baruch, M., "Optimal Correction of Mass and Stiffness Matrices Using Measured Modes," *AIAA Journal,* Vol. 20, No. 11, 1982, pp. 1623–1626.

[6]Zhang, D.-W., "The Optimal Perturbation Method for Modification of Structural Dynamic Analytical Model," *Proceedings of the International Conference on Vibration Problems in Engineering (ICVPE),* Xi'an Jiaotong Univ., Xi'an, China, 1986, pp. 672–678.

[7]Zhang, D.-W., and Zhao, L.-C., "A Minimum Norm Perturbation Method for the Modification of Structural Analytical Model," *Proceedings of the 4th International Modal Analysis Conference,* Society for Experimental Mechanics and Union College, Los Angeles, CA, Feb. 1986, pp. 541–548.

[8]Chen, S.-Y., and Fuh, J.-S., "Application of the Generalized Inverse in Structural System Identification," *AIAA Journal,* Vol. 22,

No. 12, 1984, pp. 1827, 1828.
[9]Kabe, A. M., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal,* Vol. 23, No. 9, 1985, pp. 1431–1436.
[10]Caesar, B., "Update and Identification of Dynamic Mathematical Models," *Proceedings of the 4th International Modal Analysis Conference,* Society for Experimental Mechanics and Union College, Los Angeles, CA, Feb. 1986, pp. 394–401.
[11]Kammer, D. C., "Optimum Approximation for Residual Stiffness in Linear System Identification," *AIAA Journal,* Vol. 26, No. 1, 1988, pp. 104–112.
[12]Zhang, D.-W., and Li, J.-J., "A New Method for Updating the Dynamic Mathematical Model of a Structure," Inst. of Aeroelasticity, DFVLR IB 232-87 J05, Gottingen, Germany, Dec. 1987.
[13]Wei, F. S., and Zhang, D.-W., "Mass Matrix Modification Using Element Correction Method," *AIAA Journal,* Vol. 27, No. 1, 1989, pp. 119–121.

Study of the Coupled Free Vibration of Helical Springs

W. Jiang,* T. L. Wang,† and W. K. Jones‡
Florida International University, Miami, Florida 33199

Introduction

THE study of helical spring dynamics can be dated back as early as 1890, when Michell[1] obtained three equations of motion by using the first form of Lagrange's fundamental equation. In the derivation, Michell considered displacements in three directions, but only one independent rotation. All of the moments of inertia of the typical element and the deformation produced by all forces were ignored. He showed the existence of two types of wave, producing axial or radial movement. These two waves were independent.

Love[2] obtained six equations of motion based on the same assumptions as Michell's. His equations were later modified by Yoshimura and Murata[3] to include the torsional inertia, and then by Wittrick[4] to include the rotary inertia and Timoshenko shear-deformation effects. The governing equations thus derived, however, are too difficult to permit analytical solutions. Consequently, various numerical methods, together with a variety of simplifications, were introduced to attack such problems (e.g., Mottershead,[5] Pearson and Wittrick[6]). Interested readers can refer to an extensive literature review by Pearson.[7]

Some "simple" theories were also developed along with the aforementioned "complex" theories. The simplest one treated the spring as a homogeneous rod having modified elastic and mass properties (e.g., Dick,[8] Gross,[9] Curran[10]). The coupling effects relating extension, torsion, and bending were ignored. Phillips and Costello[11] extended the simple theory to include the extensional-torsional coupling effect for springs under the action of axial forces and moments, and a set of nonlinear equations was obtained. Costello[12] further extended the investigation in the radical expansion due to axial impact, and the linear and nonlinear theories were compared in a paper by Sinha and Costello.[13]

Based on the simple theory, closed-form solutions to the coupled free vibration of helical springs have been achieved in a previous paper.[14] This Note uses examples to further explore the characteristics of the coupled vibration.

Received Dec. 3, 1990; revision received July 1, 1991; accepted for publication Aug. 4, 1991.

*Assistant Professor, Mechanical Engineering Department.
†Associate Professor, Civil Engineering Department.
‡Associate Professor, Mechanical Engineering Department.

Free Vibration of Helical Springs

It has been shown in the previous research that the static and dynamic responses of helical springs are coupled, that is, a tensile load will cause both axial and rotational displacements at the same time, and conversely, a torque will result in rotation as well as extension. Let u and θ be the axial and rotational displacement. Then the equations of motion of the spring can be expressed as

$$k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 \theta}{\partial x^2} = \gamma \frac{\partial^2 u}{\partial t^2} \tag{1a}$$

$$k_3 \frac{\partial^2 u}{\partial x^2} + k_4 \frac{\partial^2 \theta}{\partial x^2} = \mu \frac{\partial^2 \theta}{\partial t^2} \tag{1b}$$

and the free vibration solution has been found to be

$$u(x,t) = \sum_{i=1}^{2} (u_0 a_i + \theta_0 c_i)\ w_i\ (x,t) \tag{2a}$$

$$\theta(x,t) = \sum_{i=1}^{2} (u_0 b_i + \theta_0 d_i)\ w_i\ (x,t) \tag{2b}$$

where in Eqs. (1) and (2), k_1–k_4 are stiffnesses; γ is the mass and μ is the mass moment of inertia about the axis per unit length of the spring; u_0 and θ_0 are initial axial and rotational displacements; and a_i, b_i, c_i, and d_i are coefficients. All of these constants can be found in the previous paper.

The free vibration of the spring is characterized by the function

$$\omega_i(x,t) = \begin{cases} (-1)^m \left(\dfrac{t}{\omega_i} - 2m + 1\right) \\ \qquad \left(2m - 1 - \dfrac{x}{h}\right) \omega_i \le t \le \left(2m - 1 + \dfrac{x}{h}\right) \omega_i \\ (-1)^m \dfrac{x}{h} \\ \qquad \left(2m - 1 + \dfrac{x}{h}\right) \omega_i \le t \le \left(2m + 1 - \dfrac{x}{h}\right) \omega_i \end{cases} \tag{3}$$

$$m = 0, 1, \ldots$$

which generaly represents a trapezoidal wave with period $4\omega_i$, and degenerates to a triangular wave at the free end. Analysis shows that the wave function $w_1(x,t)$ characterizes the longitudinal vibration, while $w_2(x,t)$ characterizes the torsional vibration of the spring. The spring vibration is a combination of these two waves. Because of the difference in period, the resultant wave is complex and not periodic.

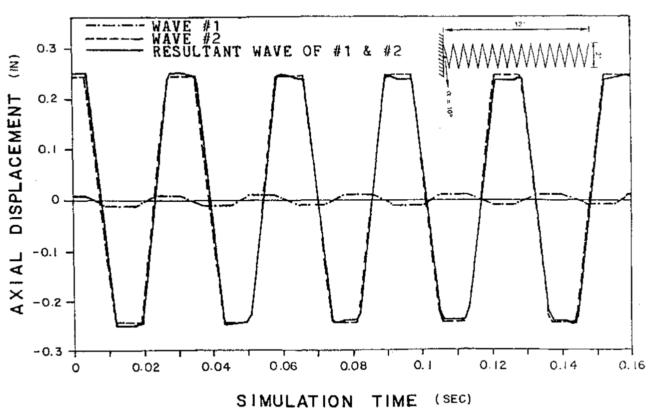

Fig. 1 Axial displacement at middle point due to wave 1, wave 2, and resultant wave of 1 and 2 for a helical angle of 10 deg and an initial axial displacement of 0.5 in.